REPORT R-345 MARCH, 1967

COORDINATED SCIENCE LABORATORY

BINARY SEQUENCE CONVOLUTIONAL MAPPING: THE CHANNEL CAPACITY OF A NON-FEEDBACK DECODING SCHEME

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This work was supported by the Joint Services Electronics Program (U. S. Army, U. S. Navy, and U.S. Air Force) under Contract DA 28 043 AMC 00073(E).

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ABSTRACT

In this paper, invertible convolutional transformations of binary sequences are examined from the point of view of performance, when the inverse transformation (decoding) is performed by a finite feed-forward transducer, which represents an approximation to the perfect feedback transducer. While this eliminates the error propagation effect, it introduces a restriction on the acceptable input sequences. The encoder-decoder system, i.e. the cascade of the direct and the inverse transducers, appears as an input-restricted noiseless channel, and a measure of performance is given by the resulting channel capacity. It is shown that as the number r of decoder stages increases, the channel capacity has an expression $C \simeq 1\text{-}Ab^r$ where the parameters $b < 1$ and A depend solely upon the structure of the set of resynchronizing states (RS-cluster) possessed by the given transformation.

1. INTRODUCTION

The sequence transformations considered in this paper are convolutional and length-preserving as performed by non-feedback binary shift-registers. Although this class of sequence transducers is less general than the class of finite state machines [1], its simplicity and flexibility make it particularly attractive in view of practical applications (e.g., in the field of cryptology).

A basic requirement is that the original sequence be uniquely reconstructible from the transformed sequence, i.e. that the transformation be invertible (information lossless).

A second problem is the implementation of the inverse transformation (decoding). In a previous paper [2] we have shown that the perfect inverse transducer (decoder) is a feedback shift-register characterized by the same Boolean function as the direct transducer (encoder). However, if the transformed sequence is perturbed by injected errors, this realization presents the error propagation effect, which is typical of convolutional feedback decoding. Since feedback is the cause of error propagation, the question arises whether non-feedback decoding is also feasible. While the general answer to this question is in the negative, because the feedforward shift register equivalent to a feedback shift register contains an infinite number of stages, we may ask ourselves what happens if the ideal infinite length decoder is "approximated" by truncation at some finite length. The analysis developed in [2] shows that in this instance some sequences are not decoded correctly, hence are ruled out at the input if correct decoding is required.

It seems reasonable, therefore, to evaluate a given sequence transformation in terms of the input constraint imposed by a given finite length decoder. Since the cascaded encoder-decoder system appears as an input-restricted noiseless channel, the evaluation can be performed in terms of channel capacity or, equivalently, in terms of entropy of the matched source.

The purpose of this paper is therefore the development of an expression for the channel capacity as a function of the decoder length and the characteristics of the transformation. This gives a quantitative formulation to the intuitive conjecture that the constraint becomes weaker as the decoder length increases. Furthermore it completes the analysis of sets of convolutional transformations: while in [2] the main interest lay on the classification of sets of Boolean functions on the basis of their cardinality ("flexibility"), the present objectives is their classification under the complementary standpoint of "performance."

2. DEFINITIONS AND REVIEW OF PREVIOUS RESULTS

The binary input sequence $\{x\} \equiv \ldots, x_{s-1}, x_s, x_{s+1}, \ldots$ is transformed into the binary sequence $\{y\} \equiv \ldots, y_{s-1}, y_s, y_{s+1}, \ldots$ according to the relation

$$y_s = x_s + f(x_{s-1}, x_{s-2}, \ldots, x_{s-n}) \qquad (1)$$

where + denotes modulo-2 addition and f is a Boolean function of n variables subject to the only condition that $f(0,0,\ldots,0) = 0$.

In the anaylsis of convolutional sequence transformations governed by relation (1) [2], we have defined $x_m | x_{m+1}$ a resynchronizing point (RP) of $\{x\}$ induced by f if and only if $y_{m+s}(s=1,2,\ldots)$ does not depend upon $x_{m-t}(t=0,1,\ldots)$: obviously the reconstructed $x_{m+s}(s=1,2,\ldots)$ does not depend upon $y_{m-t}(t=0,1,\ldots)$. We recognize that the RP's of $\{x\}$ under f coincide with the RP's of $\{y\}$ under the inverse transformation: We may therefore restrict our analysis to the direct transformation. If we call <u>irreducible</u> a segment of sequence comprised between consecutive RP's, any input sequence $\{x\}$ is a concatenation of irreducible segments: the <u>length</u> of a segment is the number of digits it contains. If r is the maximum length of the irreducible segments forming a sequence $\{x\}$, then correct decoding takes place only if the decoder contains at least (r-1) stages. The whole system is illustrated in Figure 1.

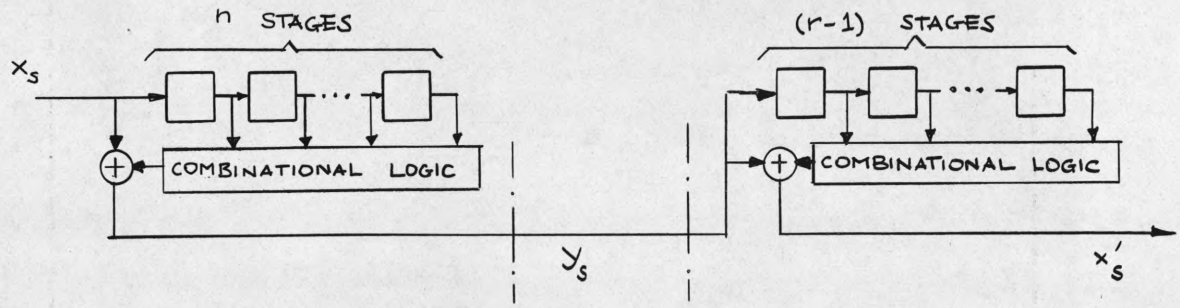

Figure 1

Convolutional Encoder and Decoder

If we now assume that for a given transformation the decoder length (r-1) is fixed, we say that a binary source is matched to the system if for any sequence $\{x\}$ generated by the source and fed to the system's input, the output sequence is the exact replica of $\{x\}$. It is therefore possible to compute the entropy of a matched source, and the upper bound of the entropy over the totality of matched sources is the channel capacity of the system.[1] Obviously, a matched source is one whose code words are irreducible segments of length at most r.

Before proceeding further, we briefly review how a given transformation function f induces RP's on a sequence $\{x\}$. It has been shown [2] that each Boolean function f of n variables possesses

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[1] This is a classical scheme. See, e.g. [4] page 8, with reference to teletype channel.

a set of "resynchronizing" n-tuples, called resynchronizing states (RS), which are all and the only n-tuples satisfying the following set of equations:

$$
\begin{aligned}
f(z_n, z_{n-1}, \ldots, z_1) &= f(0,0,\ldots,0) \\
f(\delta_1, z_n, \ldots, z_2) &= f(\delta_1, 0, \ldots, 0) \\
&\cdots \\
f(\delta_{n-1}, \ldots, \delta_1, z_n) &= f(\delta_{n-1}, \ldots, \delta_1, 0) .
\end{aligned}
$$

This set of equations establishes a pairwise association between n-tuples and leads to the definition of RS-clusters or order n, i.e. of the only admissible sets of RS's of Boolean functions of n variables. Hence, given a Boolean function f or, equivalently, its RS-cluster C, $x_m | x_{m+1}$ is an RP of $\{x\}$ under f if and only if

$$x_m = z_n,\ x_{m-1} = z_{n-1}, \ldots,\ x_{m-n+1} = z_1$$

for some $(z_n, z_{n-1}, \ldots, z_1) \in C$. Hence the RS-cluster of a given transformation defines the RP's of any sequence under this transformation. The reader is referred to [2] for a theoretical analysis of the structure of RS-clusters. For convenience we recall the following properties of RS-clusters:

1) If $(z_n, z_{n-1}, \ldots, z_1) \in C$, then $z_n = 0$
2) If $(0, z_{n-1}, \ldots, z_1) \in C$, then any right shift of $(0, z_{n-1}, \ldots, z_1)$ also belongs to C.

Given a generic n-tuple $x = (x_n, x_{n-1},\ldots,x_1)$, $(x_n, x_{n-1},\ldots,x_{n-i+1})$ and $(x_j, x_{j-1},\ldots,x_1)$ are respectively the i-digit left segment and the j-digit right segment of x.

Finally, making reference to an RS-cluster $\mathcal{C}$, let $\mathcal{F}_j$ be the set of all the binary sequences of j digits which have

i) An RP before their first digit,

ii) No RP after any of their digits.

Similarly, let $\mathcal{H}_j$ be the set of all the binary sequences of j digits which have

i) An RP before their first digit and after their j-th digit,

ii) No RP after their 1st, 2nd,...,(j-1)-th digits.

$\mathcal{H}_j$ is the set of the irreducible segments of length j. Let ν_j and σ_j be the cardinalities of $\mathcal{F}_j$ and $\mathcal{H}_j$ respectively.

As we shall see in the sequel the integer sequence $\{\sigma_j\}$ $(j = 1,2,\ldots)$ plays a fundamental role in the expression of the channel capacity of the encoder-decoder system. The following section, which is aimed at the investigation of the properties of $\{\sigma_j\}$, provides also additional insight into the structure of RS-clusters.

3. ANALYSIS OF THE SEQUENCE $\{\sigma_j\}$

For a given RS-cluster C, any binary sequence $s \in \mathcal{F}_j$ (for any positive j) can be thought of as being generated by a finite state "source" Σ, whose states $q_i (i = 1, \ldots, s; 2^{n-1} \leq s < 2^n)$ are in a one-to-one correspondence with the n-tuples $(z_1, z_2, \ldots, z_n) \in C$, and whose state diagram is a subdiagram of the diagram of the n-stage binary shift register. The source Σ can be described by its connection matrix $D = \|d_{ij}\|$, where $d_{ij} = 1$ if and only if state q_j precedes state q_i and is 0 otherwise. It is then known (see e.g. [4]) that the total number ν_j of sequences of j digits satisfies a linear recurrence relation of order at most equal to the number s of states of Σ[1].

The exponential growth of s with n may appear as a discouraging fact for the determination of the coefficients of the above-mentioned recurrence relation. Because of the structure of RS-clusters, however, there is a recurrence relation governing ν_j of order at most equal to the order n of the RS-cluster, as we shall show in the discussion that follows.

Let $C \equiv \{0, z_{n-1}, \ldots, z_1\}$ be an RS-cluster of order n, and let $\overline{C}$ be its complement in the set of all binary n-tuples. The subset $D_j \subset \overline{C}$ is the set of all the n-tuples $\{x_j, x_{j-1}, \ldots, x_1, 0, z_{n-1}, \ldots, z_{j+1}\}$, which do not belong to $D_1 \cup D_2 \cup \ldots \cup D_{j-1}$. In other words, the n-tuples

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[1] This follows from the degree of the characteristic polynomial of D and from the Cayley-Hamilton theorem.

of D_j are obtained by shifting the n-tuples of D_{j-1} one digit to the right and introducing from the left a digit x_j such that the resulting n-tuple i) is not an RS and ii) does not appear in any D_s for $s < j$. The sets D_j's are disjoint by construction. We first show that

<u>Lemma 1</u> - The sets D_j's are n in number.

<u>Proof</u>: Given any n-tuple $x = (x_n, x_{n-1}, \ldots, x_1) \in \overline{\quad}$, either x_s, $x_{s-1}, \ldots, x_1$ $(s=1,\ldots,n-1)$ is the <u>longest</u> segment which coincides with $0, z_{n-1}, \ldots, z_{n-s+1}$ for $(0, z_{n-1}, \ldots, z_1) \in$, or there is no such segment.

In the first case, for some δ_n, there is a segment $x_s, \ldots, x_1, \delta_n$ which coincides with the left (s+1)-digit segment of an RS. Therefore there is an n-tuple $(x_{n-1}, \ldots, x_1, \delta_n) \in \overline{\quad}$ (if not, the assumption on longest segment would be violated), and the longest segment of this n-tuple is of length (s+1). Iterating this argument (n-s) times, we recognize that $x=(x_n, \ldots, x_1)$ can be reached in (n-s) steps from an RS, hence $x \in D_j$ with $j \leq n-s \leq n$.

In the second case, we obtain in one step the n-tuple $(x_{n-1}, \ldots, x_1, 0)$ and argue as before with $s=1$. Q.E.D.

Let us now consider the set D_j. A <u>member</u> R_{jk} of D_j is defined as the set of n-tuples $(x_j, \ldots, x_1, 0, z_{n-1}, \ldots, z_{j+1})$ which have the same initial segment $(x_j, \ldots x_1)$. The two n-tuples $(x_j, \ldots, x_1, 0, z_{n-1}, \ldots, z_{j+1})$ and $(x_{j+1}, x_j, \ldots, x_1, 0, z_{n-1}, \ldots, z_{j+2})$ are in a <u>predecessor-successor</u> relation. Similarly, two members $R_{jk} \subset D_j$ and $R_{ih} \subset D_i$ are in a predecessor-successor relation if there are two n-tuples $x^{(1)} \in R_{jk}$ and $x^{(2)} \in R_{ih}$ such that $x^{(1)}$, $x^{(2)}$ are in a predecessor-successor relation. We can now state the following lemma.

<u>Lemma 2</u> - If $R_{jk} \subset D_j$ is a predecessor of $R_{ih} \subset D_i$, then any other member of $R_{is} \subset D_i$ has a predecessor in D_j.

<u>Proof</u>: Let $x=(x_i,\ldots,x_1,0,z_{n-1},\ldots,z_{i+1}) \in R_{ih}$ be a successor of $x'=(x'_j,\ldots,x'_1,0,z'_{n-1},\ldots,z'_{j+1}) \in R_{jk}$. If $i=j+1$ the statement is trivial because of the construction procedure of the sets $D_1,D_2,\ldots,D_n$.

If $i=j+1$, it must be $i \leq j$. Hence we have by hypothesis the following identities (digits in the same column are identical):

$$\begin{array}{llllll} x_{i-1},\ldots,x_1 & , & 0 & ,z_{n-1} & ,\ldots,z_{n-j+i},z_{n-j+i-1},z_{n-j+i-2},\ldots,z_{i+1} \\ x'_j,\ldots,x'_{j-i+2},x'_{j-i+1},x'_{j-1},\ldots,x'_1 & , & 0 & ,z'_{n-1} & ,\ldots,z'_{j+2} \end{array}$$

where $(0,z'_{n-1},\ldots,z'_{j+2})$ is the $(n-j-1)$-digit left segment of any RS. Since each n-tuple in D_j has a predecessor in D_{j-1}, by iterating the argument there is an n-tuple

$$y = (0,x'_{j-1},\ldots,x'_1,0,z'_{n-1},\ldots,z'_{j-i+2})$$

which belongs to D_{j-i+1}, such that $(0,z'_{n-1},\ldots,z'_{j-i+2})$ is the $(n-j+i-1)$-digit left segment of any RS. Now the $(n-i)$-digit left segment of y

$$(0,x'_{j-i},\ldots,x'_1,0,z'_{n-1},\ldots,z'_{j+2})$$

coincides with the $(n-i)$-digit right segment of x, i.e. with the left segment of an RS. We claim that this is the longest left segment of y which has this property. In fact, assume that $(0,x'_{j-1},\ldots,z'_{j+1})$ has this property. Then

$$(x'_{j-1},\ldots,x'_{1},0,z'_{n-1},\ldots,z'_{j})$$

which belongs by hypothesis to D_{j-1} (being a predecessor of x') would be a predecessor of

$$(x_{i-2},\ldots,x_{1},0,z_{n-1},\ldots,z_{i-1})\in D_{i-1}$$

and has therefore no successor in D_j, contrary to the hypothesis that $x\in D_j$.

This established, assume that

$$\bar{x} = (\bar{x}_{i},\ldots,\bar{x}_{1},0,z_{n-1},\ldots,z_{i+1})\in R_{i\ell} .$$

Then there is an n-tuple

$$(\bar{x}_{i-1},\ldots,\bar{x}_{1},0,x'_{j-i},\ldots,x'_{1},0,z'_{n-1},\ldots,z'_{j+1})\in D_{j}$$

which is a predecessor of $\bar{x}$ (because only after i right shifts of y the segment $(0,x'_{j-i},\ldots,z'_{j+2})$ coincides with the left segment of an RS). Q.E.D.

<u>Corollary</u> - The multiplicity of predecessors R_{jk}'s of R_{ih}'s is the same for each R_{ih}.

<u>Proof</u>: There are as many predecessors of a given R_{ih} as there are different choices of $x'_{j-i},\ldots,x'_{1}$. This obviously holds for any $R_{i\ell}\subset D_i$. Q.E.D.

We now have all the premises to prove the following

Theorem 1 - There is a recurrence relation governing the integer sequence $\{\nu_i\}$ at most of order n with integral coefficients.

Proof: By Lemma 2 and its Corollary the set of the members of D_j have as successors all the members of certain subsets D_{s_1}, $D_{s_2}, \ldots, D_{s_j}$. Hence $D_1, D_2, \ldots, D_n$ can be considered as states of a source Σ^*. Let $A = \|a_{ij}\|$ be the connection matrix of Σ^*, where a_{ij} is the multiplicity of the predecessors of D_i in D_j (each a_{ij} is a positive integer). Now, let μ_{kj} be the number of sequences generated by Σ^* having k digits and terminating in $D_j (j=1,2,\ldots,n)$, and let $\underline{u}_k = [\mu_{k1}, \ldots, \mu_{kn}]'$. Furthermore let p_j be the number members of D_j and $\underline{p}' = [p_1, \ldots, p_n]$ be a row vector: if we start with a definite n-tuple in D_1 (that is, $\underline{u}_1 = [1,0,\ldots,0]'$), then the number of different binary sequences generated by Σ having j digit is obviously given by

$$\nu_j = \underline{p}' \cdot \underline{u}_j \,.$$

Let now $h(x) = \sum_{i=o}^{n} a_i x^i$ be the characteristic polynomial of A. It follows

$$\underline{p}' \cdot h(A) \cdot \underline{u}_{j-n} = \underline{p}' \sum_{i=o}^{n} a_i \, \underline{u}_{j-n+i} = \sum_{i=o}^{n} a_i \, \nu_{j-n+1} = 0.$$

The coefficients $a_o, a_1, \ldots, a_n$ are integers because they are products of integers (and $a_n = 1$). Q.E.D.

For the determination of the coefficients $a_o, a_1, \ldots, a_n$ we proceed as follows. The integer sequence $\{\nu_j\}$ may be thought of as the "impulse response" of the source Σ. In fact assume that for $j = -1,-2,\ldots$ $\nu_j = 0$, i.e. no state of Σ is occupied; then for j=0 a transition is forced from the state $(0,0,\ldots,0)$ into $(1,0,\ldots,0)\in\Sigma$. This corresponds to assuming

$$\nu_{-n+1} = 0,\ \nu_{-n+2} = 0, \ldots,\ \nu_{-1} = 0,\ \nu_o = 1$$

as initial conditions. Application of the recurrence relation

$$\sum_{i=o}^{n} a_i \, \nu_{j-n+1} = 0 \tag{2}$$

generates the successive terms of the sequence $\{\nu_j\}$. Now the terms $\nu_1, \nu_2, \ldots, \nu_n$ for a given cluster C can be obtained by inspection, i.e. by constructing the first n levels of the binary tree T(C) of the nonresynchronizing sequences. T(C) is characterized as follows: i) each branch is labeled with the digit (0,1) produced in passing from a node of a level to a node of the successive level, where the level of a node denotes the number of branches connecting the node to the root, conventionally of level 0; ii) a transition to a node of level j is allowed if and only if the digit string identified by the path connecting the node to the root extended to the right with (n-j) 0's yields an n-tuple not belonging to C.

<u>Example</u>: For the RS-cluster or order 4

$$C \equiv (0111, 0011, 0001, 0000)$$

we have the following T(C)

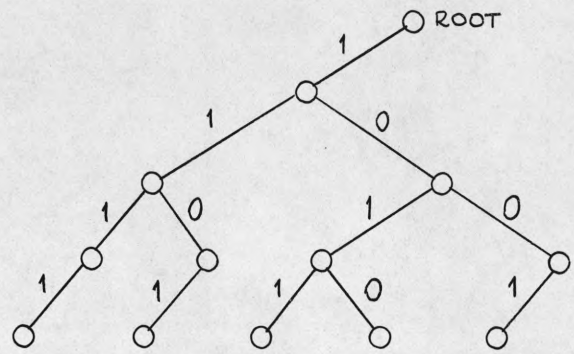

Figure 2
The diagram of the tree T(C).

from which we readily obtain $\nu_1 = 1$, $\nu_2 = 2$, $\nu_3 = 4$, $\nu_4 = 5$.

Once $\nu_1, \nu_2, \ldots, \nu_n$ are known, $a_o, a_1, \ldots, a_{n-1}$ are obtained through the following relations:

$$\begin{cases} a_{n-1} = -\nu_1 \\ a_{n-2} = -\nu_2 - a_{n-1}\,\nu_1 \\ \ldots \\ a_o = -\nu_n - a_{n-1}\,\nu_{n-1} - \cdots - a_1\,\nu_1 \,. \end{cases}$$

This completes the discussion of the sequence $\{\nu_j\}$. The results obtained are readily applicable to $\{\sigma_j\}$ by virtue of the following theorem.

<u>Theorem 2</u> - The integer sequences $\{\sigma_j\}$ and $\{\nu_j\}$ are governed by the same recurrence relation.

<u>Proof</u>: Consider the set $\mathcal{F}_j$ defined at the beginning of this section. If we extend each sequence of $\mathcal{F}_j$ by one digit (0,1), the resulting sequences of length (j+1) either have an RP after the

added digit or do not: the former constitute the set $\mathcal{H}_{j+1}$, the latter the set $\mathcal{F}_{j+1}$. Referring now to the cardinalities of $\mathcal{F}_j$, $\mathcal{F}_{j+1}$, $\mathcal{H}_{j+1}$ it follows

$$2\nu_j = \nu_{j+1} + \sigma_{j+1} . \tag{3}$$

If we now write (3) for $j = j_o+k$ for fixed j_o, multiply it by a_k and then sum side by side over $k = 0,1,\ldots,n$ we obtain

$$\sum_{k=o}^{n} a_k \nu_{j_o+k} = \sum_{k=o}^{n} a_k \nu_{j_o+k+1} + \sum_{k=o}^{n} a_k \sigma_{j_o+k+1} .$$

Since $\sum_{k=o}^{n} a_k \nu_{j_o+k} = \sum_{k=o}^{n} a_k \nu_{j_o+k+1} = 0$ by hypothesis, it follows that

$$\sum_{k=o}^{n} a_k \sigma_{j_o+1+k} = 0 \qquad \text{Q.E.D.} \tag{2'}$$

The sequence $\{\sigma_j\}$ can be obtained from $\{\nu_j\}$ through relation (3).

<u>Example</u> - Referring to the same RS-cluster considered in the previous example we have

$\{j\}$	...	-4,	-3,	-2,	-1,	0,	1,	2,	3,	4,	5,	6,...
$\{\nu_j\}$	...	0,	0,	0,	0,	1,	1,	2,	4,	5,	9,	14,...
$\{\sigma_j\}$	...	0,	0,	0,	0,	0,	1,	0,	0,	3,	1,	4,...

We conclude this section with a rather important theorem on the roots of $h(x)$.

<u>Theorem 3</u> - The polynomial $h(x) = x^n + a_{n-1} x^{n-1} + \ldots + a_o$ has

a positive root $1 \leq \xi_1 < 2$, which is not exceeded in modulus by any other root of h(x).

Proof: Consider the previously introduced matrices D and A, i.e. the connection matrices of the sources Σ and Σ^*, respectively. Both A and D are non-negative matrices ($a_{ij} \geq 0$, $d_{hk} \geq 0$). Hence from the Perron-Frobenius theorem for non-negative reducible matrices (see [5], p. 66) D (A) has a non-negative eigenvalue η_1 (ξ_1) such that the moduli of all the eigenvalues of D (A) do not exceed η_1 (ξ) Let $p(x) = x^s + p_{s-1} x^{s-1} + \ldots + p_o$ be the characteristic polynomial of D. Since ν_j satisfies simultaneously

$$\nu_j = - (a_{n-1} v_{j-1} + \ldots + a_o v_{j-n})$$

$$\nu_j = - (p_{s-1} v_{j-1} + \ldots + p_o v_{j-s})$$

and the asymptotic expression of ν_j depends solely upon the highest positive root of h(x) = 0 or of p(x) = 0 (and its multiplicity), we conclude that $\xi_1 = \eta_1$. We may therefore restrict our analysis to η_1.

Again by the Perron-Frobenius theorem the eigenvector $\underline{v}$ of D corresponding to η_1 is non-negative, i.e.

$$D \underline{v} = \eta_1 \underline{v} \qquad (\underline{v} \geq 0, \underline{v} \neq 0) . \qquad (4)$$

Because of the nature of a subdiagram of a shift register diagram, each column of D contains either one or two elements equal to 1, all the others being 0 (but not all columns contain two elements equal to 1). Therefore if we premultiply (4) by the row vector $\underline{u}' = [1,1,\ldots,1]$

we have

$$\underline{u}' D \underline{v} = \eta_1 \underline{u}' \underline{v} .$$

But $\underline{u}'D = \underline{u}' + \underline{w}'$ where $\underline{w}' = [w_1, w_2, \ldots, w_s]$ is a non-negative vector whose components are either 0 or 1 (but not all simultaneously 1). Hence

$$\underline{u}' \underline{v} + \underline{w}' \underline{v} = \eta_1 \underline{u}' \underline{v} .$$

Since $\underline{v} \geq 0$, $\underline{v} \neq 0$, $\underline{w}' \geq 0$ we have $\underline{u}' \underline{v} > 0$ and $\underline{w}' \underline{v} \geq 0$. Therefore

$$\underline{u}' \underline{v} \leq \eta_1 \underline{u}' \underline{v} .$$

Dividing by $\underline{u}' \underline{v} > 0$, we have $\eta_1 \geq 1$. We now claim that $\underline{u}' \underline{v} > \underline{w}' \underline{v}$. If D is irreducible, the statement is trivial since $\underline{v} > 0$. If D is reducible, consider the states of Σ divided in recurrent and transient, according to the Markov chain terminology. Obviously, there is at least one recurrent state. Further, there is at least one recurrent state with only one subsequent state, otherwise Σ could generate a sequence of n 0's (which is an RS): if this is state q_j, $w_j = 0$. State q_j belongs to some irreducible subset of states: If $v_j = 0$, since $\underline{v}$ is non-negative and $\eta_1 \geq 1$, v_s is 0 for every other state q_s of the subset. It can also be shown that $v_s = 0$ for every state of a chain of states originating from a state with no predecessor. We see that the assumption that $v_j = 0$ for every recurrent state j for which $w_j = 0$ leads to the conclusion $\underline{v} = 0$, thus violating the Perron-Frobenius theorem. Hence for some j, $w_j = 0$, $v_j > 0$, which proves $\underline{u}'\,\underline{v} > \underline{w}'\,\underline{v}$. It follows that

$$\underline{u}'\,\underline{v} + \underline{u}'\,\underline{v} > \eta_1\,\underline{u}'\,\underline{v}$$

or $\eta_1 < 2$.

4. THE CHANNEL CAPACITY

To obtain the upper bound of the entropy of a source m matched to the system, i.e. the channel capacity, we must determine the number of sequences of u digits, for arbitrary u, which are admitted at the system's input. Since we are interested in asymptotic results for $u \to \infty$, without loss of generality we may assume that the initial digit of each admissible sequence occurs immediately after an RP.

With this premise, let r be the bound on the maximum length irreducible segment. For $j \leq r$, σ_j is the number of irreducible segments of length j. Similarly, let τ_j be the number of admissible sequences of length j which do not contain a RP after each of their digits: obviously $\tau_j = \nu_j$ for $j \leq r-n$, but generally $\tau_j \leq \nu_j$ for $r-n < j \leq r$. With ρ_j we denote

$$\rho_j = \tau_j + \sigma_j \qquad (j = 1,2,\dots,r) .$$

Let S_u be the number of admissible sequences of u digits such that $x_u | x_{u+1}$ is a RP, i.e. comprised between two RP's, generally not consecutive. S_u is given by the following linear recurrence relation

$$S_u = \sigma_1 S_{u-1} + \sigma_2 S_{u-2} + \dots + \sigma_r S_{u-r} . \tag{5}$$

The total number of admissible sequences of u digits is denoted with T_u and is given by

$$T_u = S_u + \rho_1 S_{u-1} + \dots + \rho_r S_{u-r} \;. \tag{6}$$

We can now show that

<u>Theorem 3</u> - The variables S_u and T_u are governed by the same recurrence relation.

<u>Proof</u>: We write equation (6) for $T_u, T_{u-1}, \dots, T_{u-r}$ $(u > 2r)$. In matrix form we have

$$\begin{bmatrix} S_{u-1} & S_{u-2} & \dots & S_{u-r} \\ S_{u-2} & S_{u-3} & \dots & S_{u-r-1} \\ & \dots & & \\ S_{u-r-1} & S_{u-r-2} & \dots & S_{u-2r} \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_r \end{bmatrix} = \begin{bmatrix} T_u \\ T_{u-1} \\ \vdots \\ T_{u-r} \end{bmatrix} - \begin{bmatrix} S_u \\ S_{u-1} \\ \vdots \\ S_{u-r} \end{bmatrix}$$

We now notice that each column of the matrix as well as the column vector $[S_u, S_{u-1}, \dots, S_{u-r}]$ are orthogonal to the row vector $[1, -\sigma_1, -\sigma_2, \dots, -\sigma_r]$, since the corresponding inner products yield equation (5) for $S_u, S_{u-2}, \dots, S_{u-r}$. Hence by premultiplying the previous equation by $[1, -\sigma_1, -\sigma_2, \dots, -\sigma_r]$ we obtain

$$T_u - \sigma_1 T_{u-1} - \sigma_2 T_{u-2} - \dots - \sigma_r T_{u-r} = 0$$

which proves our assertion. Q.E.D.

It follows that the asymptotic approximations [3] of T_u and S_u differ only by a multiplicative constant, i.e. the two variables have the same "rate of growth." Hence we may limit our analysis to S_u. Clearly the equation

$$x^r - \sigma_1 s^{r-1} - \ldots - \sigma_r = 0 \tag{5'}$$

has one and only one real solution in the interval $(0,\infty)$: in fact, because $\sigma_j \geq 0$ for $j = 1,2,\ldots$, the function

$$q(w) = \frac{\sigma_1}{w} + \frac{\sigma_2}{w^2} + \ldots \frac{\sigma_r}{w^r}$$

is monotonically decreasing from $+\infty$ to 0 as w increases from 0 to $+\infty$. Hence $q(w) = 1$ for one and only one value of w, i.e. (5') has only one positive solution w_1. It follows that the asymptotic expression of S_u is (see e.g. [4], p. 8)

$$S_u \sim c\, w_1^u \tag{7}$$

where c is a constant.

We must now determine the value w_1. For $r \to \infty$, clearly all sequences are admissible and $w_1 \to 2$: hence we take 2 as "initial guess" of w_1 and apply the Newton-Raphson approximation. Specifically

$$w_1 \simeq 2 - \frac{f(2)}{f'(2)} . \tag{8}$$

In order to attain a weak input constraint r must be conveniently large; it follows that $(2-w_1)$ is rather small and therefore one iteration of relation (8) is generally sufficient. With this assumption, we write after trivial manipulations:

$$w_1 \simeq 2 \left(1 - \frac{1}{r + \dfrac{\sum_{1}^{r} j\sigma_j 2^{-j}}{1 - \sum_{1}^{r} \sigma_j 2^{-j}}} \right) \tag{8'}$$

and the channel capacity is given by [4]

$$C = \log_2 w_1$$

where C is expressed in bit/binit. Relation (8') does not explicitly show the dependence of C upon r for a given RS-cluster $\mathcal{C}$. This linkage, however, can be obtained through the known structure of the sequence $\{\sigma_j\}$. In Appendix I we show that the generating function of the sequence $\{\sigma_j 2^{-j}\}$, i.e.

$$G(z) = \sum_{j=1}^{\infty} \sigma_j 2^{-j} z^j$$

is a rational function of z, that $G(1) = 1$ and that

$$G'(1) = \sum_{j=1}^{\infty} j\sigma_j 2^{-j} = \frac{2^n}{h(2)}$$

where $h(x) = x^n + a_{n-1} x^{n-1} + \ldots + a_o$ is the characteristic polynomial of the recurrence relation (2) governing both $\{\nu_j\}$ and $\{\sigma_j\}$.

With these results, relation (8') becomes

$$w_1 \simeq 2\left(1 - \frac{\sum_{j=r+1}^{\infty} \sigma_j 2^{-j}}{r\sum_{j=r+1}^{\infty} \sigma_j 2^{-j} + G'(1) - \sum_{j=r+1}^{\infty} j\sigma_j 2^{-j}}\right) \tag{8''}$$

For large r, an approximation to w_1 is obtained if we replace σ_j with its asymptotic expression. The search for such an expression is performed in Appendix II, where it is shown that

$$\sigma_j \simeq b\, j^{m-1}\, \xi_1^{\,j} \tag{9}$$

$$b = \frac{\xi_1^{\,n-m}}{g^{(m-1)}(\xi_1)}\left(\frac{2}{\xi_1} - 1\right). \tag{10}$$

Here $1 \leq \xi_1 < 2$ is the highest positive root of $h(x) = 0$ (see Theorem 3), m its multiplicity, and $g^{(m-1)}(x)$ is the (m-1)-th derivative with respect to x of $g(x) = h(x)/(x - \xi_1)$.

We now introduce approximation (9) into (8"), restricting ourselves for simplicity to the case m = 1. Although the general case m > 1 presents no conceptual difficulty, it is no explicitly treated here because its formal complications do not shed additional light on the implications of relation (8"). Therefore, for m = 1 we have

$$w_1 \simeq 2\left(1 - \frac{1}{\frac{2 - \xi_1}{b \cdot h(2)}\, 2^{n-1} \left(\frac{2}{\xi_1}\right)^{r+1} - \frac{2}{2 - \xi_1}}\right) \tag{8'''}$$

We now notice that the second term in the denominator of the fraction is a constant, while the first grows geometrically with ratio $\frac{2}{\xi_1} > 1$; Hence, for large r the second term may be neglected. By substituting (10) into (8''') we have

$$w_1 \sim 2 \left(1 - \frac{h(2)}{\xi_1 \cdot g(\xi_1)} \cdot \left(\frac{\xi_1}{2}\right)^{n+r}\right).$$

Finally, since for $x << 1$, $\log_e (1+x) \simeq x$ we have

$$C = \log_2 w_1 \simeq 1 - \log_2 e \cdot \frac{h(2)}{\xi_1 \cdot g(\xi_1)} \left(\frac{\xi_1}{2}\right)^{n+r} \qquad (11)$$

which represents the desired relation between C and r, when the inverse transducer contains (r-1) stages. We close this section with an application of relation (11).

<u>Example</u>: Consider the RS-cluster $C \equiv$ (01000, 01001, 01010, 00100, 00101, 00010, 00001, 00000, 00110, 00011) of order n=5. Application of the procedure described in Section 3 gives the following difference equation for ν_j.

$$\nu_j = \nu_{j-1} + \nu_{j-3} + 2\nu_{j-5}$$

or, equivalently,

$$h(x) = x^5 - x^4 - x^2 - 2 .$$

The highest positive root of $h(x) = 0$ is $\xi_1 \simeq 1.644$, (simple). Furthermore

$$g(x) = x^4 + 0.644x^3 + 1.058x^2 + 0.739x + 1.214 .$$

From $h(2) = 10$, $g(\xi_1) = 15.499$, we have

$$C = 1 - 1.441\times \frac{10}{1.644\times 15.449} \cdot (0.822)^{n+r} = 1 - 0.568\times(0.822)^{n+r}$$

5. CONCLUSIONS

The expression obtained in the previous section for the channel capacity of an encoder-decoder cascade for non-feedback convolutional mapping of binary sequences represents a quantitative tool for the comparative evaluation of RS-clusters. In fact, the choice of an RS-cluster for the transformation of binary sequences may depend, in practical cases, on the two following considerations:

A) The "flexibility" of the RS-cluster, i.e. the cardinality of the set of functions (a vector space) possessing the cluster; and

B) The "input restriction" of the encoder-decoder scheme based on the RS-cluster, i.e. a measure of the limitation imposed on the input sequences by the adoption of a finite feedforward inverse transducer.

While the first problem was tackled in [2], the purpose of this paper is to close the gap and to provide a theoretical scheme for the evaluation of RS-clusters both from the point of view of "richness" and that of "goodness."

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Appendix I

THE GENERATING FUNCTION G(z) OF $\{\sigma_j 2^{-j}\}$

Let us consider the expression $\sum_{j=1}^{\infty} \sigma_j 2^{-j}$, i.e. the sum of the terms of the sequence $\{\sigma_j 2^{-j}\}$. If we multiply the recurrence relation

$$\sum_{k=0}^{n} a_k \, \sigma_{j+k} = 0 \tag{2'}$$

by $\left(\frac{z}{2}\right)^j$ (z is real-valued) and sum over $j=1,2,\ldots$, we obtain

$$\sum_{j=1}^{\infty} \sum_{k=0}^{n} \left(\frac{z}{2}\right)^{-k} a_k \, \sigma_{j+k} \left(\frac{z}{2}\right)^{j+k} = \sum_{k=0}^{n} a_k \left(\frac{z}{2}\right)^{-k} \sum_{j=1}^{\infty} \sigma_{j+k} \left(\frac{z}{2}\right)^{j+k} = 0$$

or

$$0 = \sum_{k=0}^{n} a_k \left(\frac{z}{2}\right)^{-k} \left[\sum_{j=1}^{\infty} \sigma_j 2^{-j} z^j - \sum_{j=1}^{k} \sigma_j \left(\frac{z}{2}\right)^j\right] . \tag{12}$$

We recognize that the function

$$G(z) = \sum_{j=1}^{\infty} \sigma_j 2^{-j} z^j$$

is the generating function of the sequence $\{\sigma_j 2^{-j}\}$. From (12) we readily have

$$G(z) = \frac{\sum_{k=0}^{n-1} \left(\frac{2}{z}\right)^k \sum_{j=1}^{n-k} a_{k+j} \, \sigma_j}{\sum_{k=0}^{n} a_k \left(\frac{2}{z}\right)^k} .$$

It is shown in Section 3 that

$$\nu_i = -a_{n-1}\,\nu_{i-1} - \dots - a_{n-i+1}\,\nu_1 - a_{n-i}\,\nu_o \qquad (i=1,2,\dots,n)$$

which, combined with relation (3), yields

$$a_n\sigma_1 = -a_{n-1}$$

$$\sum_{j=1}^{i} a_{n-i+j}\,\sigma_j = a_{n-i} \qquad (i=2,3,\dots,n)\ .$$

Now, after replacing i with (n-k) in this expression (k=0,1,...,n-2), relation (15) becomes

$$G(z) = \frac{-\,a_{n-1}\left(\frac{2}{z}\right)^{n-1} + \sum_{k=0}^{n-2} a_k\left(\frac{2}{z}\right)^k}{\sum_{k=0}^{n} a_k\left(\frac{2}{z}\right)^k}\ .$$

Since $a_n = 1$, $a_{n-1} = -1$, $\lim_{z\to 1} G(z) = \sum_{j=1}^{\infty} \sigma_j 2^{-j} = 1$. Furthermore

$$\lim_{z\to 1} G'(z) = \sum_{j=1}^{\infty} j\sigma_j 2^{-j} = \lim_{z\to 1} \frac{\frac{d}{dz}\left[-a_{n-1}\left(\frac{2}{z}\right)^{n-1} + \sum_{k=0}^{n-2} a_k\left(\frac{2}{z}\right)^k - \sum_{k=0}^{n} a_k\left(\frac{2}{z}\right)^k\right]}{\sum_{k=0}^{n} a_k\left(\frac{2}{z}\right)^k}$$

$$= \frac{2^n}{h(2)}$$

where $h(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_o$ is the characteristic polynomial of the recurrence relation (2).

Appendix II

DETERMINATION OF AN ASYMPTOTIC EXPRESSION FOR THE SEQUENCE $\{\sigma_j\}$

As previously noted, $\{\sigma_j\}$ is given by

$$\sigma_j = 2\nu_{j-1} - \nu_j$$

and $\{\nu_j\}$ satisfies the recurrence relation

$$\sum_{k=0}^{n} a_k \nu_{j-k} = 0 \qquad (j=1,2,\ldots) \tag{2}$$

with $\nu_o = 1$, $\nu_{-1} = 0$, ... $\nu_{-n+1} = 0$ as initial conditions. We aim to find an asymptotic expression for $\{\nu_j\}$, i.e. of ν_j for very large j.

Let

$$h(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1$$

be the characteristic polynomial of the recurrence relation (2), and let $\xi_1, \xi_2, \ldots, \xi_s$ $(s \leq n)$ be the roots of $h(x) = 0$ and m_s the multiplicity of ξ_s. It is well-known (see e.g. [3]) that the general solution of (2) is of the form:

$$\nu_j = \sum_{i=1}^{s} (b_{i,o} + \ldots + b_{i,m_i-1}\, j^{m_i-1})\, \xi_i^{\,j} .$$

From Theorem 3, we know that $1 \leq \xi, < 2$ is the highest positive root of $h(x)$. For simplicity let $m_1 = m$. Asymptotically we have

$$\nu_j \simeq b_{1,m-1}\, j^{m-1}\, \xi_1^{\,j} .$$

Our task is now to determine the constant $b_{1,m-1}$. We assume for simplicity that $\xi_2,\ldots,\xi_s$ are simple roots of $h(x) = 0$; extension to the general case follows almost immediately. The column vector $[b_{1,m-1}, b_{1,m-2},\ldots, b_{1,0}, b_{2,0},\ldots, b_{s,0}]$ satisfies the following equation

$$\begin{bmatrix} 0 & 0 & \cdots & 1 & 1 & \cdots \\ (-1)^{m-1}\xi_1^{-1} & (-1)^{m-2}\xi_1^{-1} & \cdots & \xi_1^{-1} & \xi_2^{-1} & \cdots\ \xi_s^{-1} \\ (-2)^{m-1}\xi_1^{-2} & (-2)^{m-2}\xi_1^{-2} & \cdots & \xi_1^{-2} & \xi_2^{-2} & \cdots\ \xi_s^{-2} \\ \cdot & \cdot & & \cdot & & \cdot \\ (-n+1)^{m-1}\xi_1^{-n+1} & (-n+1)^{m-2}\xi_1^{-n+1} & \cdots & \xi_1^{-n+1} & \xi_2^{-n+1} & \cdots\ \xi_s^{-n+1} \end{bmatrix} \begin{bmatrix} b_{1,m-1} \\ b_{1,m-2} \\ \vdots \\ b_{1,0} \\ b_{2,0} \\ \vdots \\ b_{s,0} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where the column vector $[1,0,\ldots,0,0,\ldots,0]$ represents the given initial conditions. If we replace the first column of the given matrix with the vector $[1, x^{-1}, x^{-2}, \ldots, x^{-n+1}]$, the matrix determinant is a polynomial $\Delta(x^{-1})$ of degree $(n-1)$ in the variable x^{-1}. We notice the following facts:

1) for $x = \xi_1, \xi_2, \xi_3, \ldots, \xi_s$, $\Delta(x^{-1}) = 0$, since the matrix has two equal columns.

2) The polynomial

$$p_k(x^{-1}) = x\frac{d}{dx}\overbrace{\left(\ldots\left\{x\frac{d}{dx}\left[x\frac{d\Delta(x^{-1})}{dx}\right]\right\}\ldots\right)}^{k \text{ parentheses}}$$

for $k = 1,2,\ldots,m-2$, possesses ξ_1^{-1} as a root (for the same reason as in 1)). This means that ξ_1^{-1} is a root of $\frac{d\Delta(x^{-1})}{dx}$, $\frac{d^2\Delta(x^{-1})}{dx^2}$, ..., $\frac{d^{m-2}\Delta(x^{-1})}{dx^{m-2}}$. We conclude that $\Delta(x^{-1})$ has $\xi_2^{-1}, \ldots, \xi_s^{-1}$ as simple roots and ξ_1^{-1} as a root of multiplicity $(m-1)$. Since the monic polynomial $g(x) = h(x)/(x-\xi_1)$ has $\xi_2, \ldots, \xi_s$ as simple roots and ξ_1 as a root of multiplicity $(m-1)$ we say that

$$\Delta(x^{-1}) = c\,\frac{g(x)}{x^{n-1}\,\xi^{m-1}\,\xi_2\cdots\xi_s} \tag{13}$$

where c is a constant. The determinant Δ of the matrix is given by

$$\Delta = p_{m-1}(\xi_1^{-1}).$$

Since $\Delta(\xi_1^{-1}) = 0$ and $\left|\frac{d^k\Delta(x^{-1})}{dx^k}\right|_{x=\xi_1} = 0$ for $k = 1,2,\ldots, m-2$ we have

$$\Delta = \xi_1^{m-1} \left| \frac{d^{m-1}\Delta(x^{-1})}{dx^{m-1}} \right|_{x=\xi_1}$$

or, equivalently from (13) (since also $g(\xi_1) = g'(\xi_1) = \ldots = g^{(m-2)}(\xi_1) = 0$)

$$\Delta = \xi_1^{m-1} \cdot \frac{c}{\xi_1^{m-1} \cdot \xi_2 \cdots \xi_s} \xi_1^{-n+1} g^{(m-1)}(\xi_1)$$

$$= \frac{c\, g^{(m-1)}(\xi_1)}{\xi_1^{n-1} \xi_2 \cdots \xi_s} .$$

The constant $b_{1,m-1}$ can be obtained by application of the Cramer's rule, i.e.

$$b_{1,m-1} = \frac{\Delta_1}{\Delta}$$

where Δ_1 is the cofactor of the term at the intersection of the first row and first column in the system's matrix. This cofactor is the constant term in $\Delta(x^{-1})$, that is (because $g(x)$ is monic)

$$\Delta_1 = \frac{c}{\xi^{m-1} \cdot \xi_2 \cdots \xi^s} .$$

Finally we have

$$b_{1,m-1} = \frac{\xi_1^{n-m}}{g^{(m-1)}(\xi_1)} .$$

Returning now to the sequence $\{\sigma_j\}$, and recalling relation (3), we have

$$\sigma_j = \Sigma \left\{ \frac{2}{\xi_i} \left[b_{io} + \ldots + b_{i,m_i-1}(j-1)^{m_i-1} \right] - \left[b_{i,o} + \ldots\ b_{i,m_i-1} j^{m_i-1} \right] \right\} \xi_i^{\,j}$$

namely the coefficient of $\xi_i^{\,j}$ is a polynomial of degree (m_i-1) in the variable j. Obviously, the coefficient of j^{m_i-1} is

$$b_{i,m_i-1} \left(\frac{2}{\xi i} - 1\right) .$$

It follows immediately that the asymptotic expression of σ_j is

$$\sigma_j \simeq \frac{\xi_1^{\,n-m}}{g^{(m-1)}(\xi_1)} \left(\frac{2}{\xi_1} - 1\right) j^{m-1}\ \xi_1^{\,j} .$$

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3. REPORT TITLE

BINARY SEQUENCE CONVOLUTIONAL MAPPING: THE CHANNEL CAPACITY OF A NON-FEEDBACK DECODING SCHEME

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

5. AUTHOR(S) (Last name, first name, initial)

Preparata, Franco P.

6. REPORT DATE	7a TOTAL NO. OF PAGES	7b. NO. OF REFS.
March, 1967	33	5

8a. CONTRACT OR GRANT NO. DA 28 043 AMC 00073(E) b. PROJECT NO. 20014501B31F c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) R-345 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. AVAILABILITY/ LIMITATION NOTICES

Distribution of this report is unlimited

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Joint Services Electronics Program thru U. S. Army Electronics Command Ft. Monmouth, New Jersey 07703

13. ABSTRACT

In this paper, invertible convolutional transformations of binary sequences are examined from the point of view of performance, when the inverse transformation (decoding) is performed by a finite feed-forward transducer, which represents an approximation to the perfect feedback transducer. While this eliminates the error propagation effect, it introduces a restriction on the acceptable input sequences. The encoder-decoder system, i.e. the cascade of the direct and the inverse transducers, appears as an input-restricted noiseless channel, and a measure of performance is given by the resulting channel capacity. It is shown that as the number r of decoder stages increases, the channel capacity has an expression $C \simeq 1-Ab^r$ where the parameters $b < 1$ and A depend solely upon the structure of the set of resynchronizing states (RS-cluster) possessed by the given transformation.

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KEY WORDS	LINK A		LINK B		LINK C	
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